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A Counterexample to a Conjecture of Hasson

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Let $f \in C_{[a,b]}$ and denote by π_n the class of all algebraic polynomials of degree $\leqslant n$. Define $E_n(f) = \min_{p \in \pi_n} \|f - p\|$ and $E_n^k(f) = \min\{\|f - p\|: p \in \pi_n \text{ and } p^{(k)}(0) = 0\}$ where $\|h\| = \max_{a \leqslant x \leqslant b} |h(x)|$ for $h \in C_{[a,b]}$. Hasson in [1] proved that, for $k \geqslant 1$,

$$\lim_{n \to \infty} \frac{E_n^k(f)}{E_n(f)} = \infty$$

if $f \in C_{[a,b]}^k$ and $f^{(k)}(0) \neq 0$ where $a < 0 < b$, or if $f \in C_{[a,b]}^{2k}$ and $f^{(k)}(0) \neq 0$ where $ab = 0$. Also, in that paper Hasson conjectured that if $f \in C_{[-1,1]}$ and f' does not exist at some interior point of $[-1, 1]$, then

$$\overline{\lim_{n \to \infty}} \frac{E_n^k(f)}{E_n(f)} < \infty. \tag{1}$$

In this paper, we construct a counterexample to show this conjecture is false.

Define $f(x) = \sum_{n=1}^{\infty} (1/a_n) \cos a_n \theta$ with $x = \cos \theta$, $\theta \in [0, \pi]$ where $\{a_n\}_{n=1}^{\infty}$ are all odd positive integers satisfying

$$\frac{a_{n+1}}{a_n} = 4r_n + 1 \tag{2}$$

with each r_n a positive integer and

$$\sum_{j=n+1}^{\infty} \frac{1}{a_j} \leqslant \frac{1}{a_n^{k+1}}. \tag{3}$$

It is clear that $f(x) \in C_{[-1,1]}$ and

$$\frac{a_{n+m}}{a_n} = \prod_{j=n}^{n+m-1} (4r_j + 1) = 4p + 1$$

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for some positive integer p and $E_n(f)=\sum_{a_j>n}^{\infty} 1/a_j$ (see [3]). Furthermore,

$$E_{a_n}(f)=\sum_{j=n+1}^{\infty}\frac{1}{a_j}\leqslant\frac{1}{a_n^{k+1}}.$$

Next we show $f'(x)$ does not exist at $x=\cos(\pi/2a_1)$. Let $h_m=\pi/4a_m\to 0$. Then, for $m\geqslant 2$,

$$\begin{aligned}&\left|\frac{f[\cos(\pi/2a_1+h_m)]-f(\cos(\pi/2a_1))}{h_m}\right|\\&=\left|\frac{1}{h_m}\left\{\sum_{j=1}^{m}\frac{1}{a_j}\left[\cos a_j\left(\frac{\pi}{2a_1}+h_m\right)-\cos\frac{a_j\pi}{2a_1}\right]\right.\right.\\&\quad\left.\left.+\sum_{j=m+1}^{\infty}\frac{1}{a_j}\cos\left(\frac{a_j\pi}{2a_1}+a_jh_m\right)\right\}\right|\\&\geqslant\sum_{j=1}^{m}\sin\xi_j-\frac{4a_m}{\pi}\sum_{j=m+1}^{\infty}\frac{1}{a_j}\geqslant\frac{m}{2}-1\end{aligned}$$

since $\xi_j\in(\pi/2,(3/4)\pi)$ by the mean value theorem applied to the first summation.

This shows $f'(x)$ does not exist at $x=\cos(\pi/2a_1)\in(-1,1)$. But

$$\frac{E_n^k(f)}{E_n(f)}\leqslant\frac{\|P_n-P_n^k\|-\|f-P_n\|}{E_n(f)}\geqslant\frac{|b_k^{(n)}|\,E_n^k(x^k)}{E_n(f)}-1,$$

where $\|P_n-f\|=E_n(f)$, $\|P_n^k-f\|=E_n^k(f)$, and $b_k^{(n)}$ is the coefficient of x^k in P_n. We can assume $b_k^{(a_n)}$ does not go to zero; otherwise, we can take $f(x)+x^k$ instead of $f(x)$.

By Theorem 2.5 of [1], $E_n^k(x_k)\geqslant N_k/n^k$ with N_k independent of n. Thus,

$$\overline{\lim_{n\to\infty}}\frac{E_n^k(f)}{E_n(f)}\geqslant\overline{\lim_{n\to\infty}}\frac{E_{a_n}^k(f)}{E_{a_n(f)}}\geqslant\overline{\lim_{n\to\infty}}\frac{(b_k^{(a_n)})(N_k/a_n^k)}{1/a_n^{k+1}}=\infty.$$

Remark. We can show even more: For any sequence B_n with $\lim_{n\to\infty}B_n=\infty$, there exists a function $f\in C_{[-1,1]}$ such that $f'(x)$ does not exist at some point in $(-1,1)$, and

$$\overline{\lim_{n\to\infty}}\frac{E_n^k(f)}{E_n(f)\,B_n}=\infty.$$

To show this we only have to alter (3) to $\sum_{j=n+1}^{\infty}1/a_j\leqslant 1/B_na_n^{k+1}$ in constructing the above function f. Also this example with little change can be applied to $C_{[a,b]}$ with $0\in[a,b]$.

Since the above conjecture is not true, one is led to inquire about the lim inf in place of the lim sup in quotient (1). This remains open. In [2] M. Hasson and O. Shisha proved the following theorem:

THEOREM A. *Let $a<0<b$, let $k>0$ be an integer, let $0<\alpha<1$, and suppose for some positive A_k, $E_n^k(f)/E_n(f) \geqslant A_k n^\alpha$ holds. Then $f^{(k)}$ exists in (a, b) and on each $[a', b']$ with $a<a'<b'<b$, $f^{(k)}$ satisfies a Lipschitz condition of order α.*

This theorem implies that if $f^{(k)}(x)$ does not exist at some point in (a, b), then for any $\delta>0$

$$\varliminf_{n \to \infty} \frac{E_n^k(f)}{E_n(f)\, n^\delta} = 0. \tag{4}$$

For $f \in C_{[a,b]}$ and $ab=0$, using similar techniques together with the estimate of $E_n^k(x^k)$, $E_n^k(x^k) \leqslant N_k/n^{2k}$ of [1], we can prove

THEOREM B. *Let $ab=0$, let $k>0$ be an integer, let $0<\alpha<1$, and suppose form some positive A_k, $E_n^k(f)/E_n(f) \geqslant A_k n^\alpha$ holds. Then $f^{(2k)}$ exists in (a, b) and on each $[a', b']$ with $a<a'<b'<b$, $f^{(2k)}$ satisfies a Lipschitz condition of order α.*

This also implies (4) for any $\delta>0$ if $f^{(2k)}(x)$ does not exist at some point in (a, b) for $ab=0$.

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